THE
MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VII. JANUARY, 1914. No. 109.

INTERNATIONAL CONFERENCE ON MATHEMATICAL TEACHING.

PARIS, APRIL 1-4, 1914.

THE International Commission on Mathematical Teaching will meet at the Sorbonne, Paris, on April 1, 2, 3, 4, in order to discuss two questions: (A) The results obtained by the introduction of the Calculus into the higher classes of Secondary Schools; (B) The place of Mathematics in Higher Technical Education.

PROGRAMME. April 1. Afternoon—Meetings of the Central Committee and of the Delegates. Evening—Meeting of the French Mathematical Society at the Sorbonne.

April 2. Morning—General Opening Meeting. Chairman—M. Lucien Poincaré, Director of Secondary Education. Speeches by Prof. P. Appell, Prof. F. Klein, Prof. Emile Borel and Prof. d'Ocagne. Afternoon—Discussion of Question (A).

April 3. Morning and Afternoon—Discussion of Question (B). Evening—Meeting of the French Society of Civil Engineers.

April 4. Morning—Discussion of Questions (A) and (B) (continued). Afternoon—Meeting of Delegates. Evening—Reception by H.H. Prince Bonaparte.

The British Committee consists of Sir George Greenhill, Prof. E. W. Hobson and Mr. C. Godfrey.

Conference Tickets. British applications for tickets (which are gratis) should be made *before February* 26, 1914, to Mr. H. D. Ellis, 12 Gloucester Terrace, Hyde Park, London, W., from whom a programme may be obtained.

A reduction of 50 per cent. on the French railways is made to Members of the Conference. Full particulars will be sent with the Conference Tickets.

Other Meetings (for which members of the Conference may obtain tickets). The French Philosophical Society will hold meetings on April 6, 7, 8, at which questions of mathematical philosophy will be discussed.

The French Physical Society will hold its annual meeting and exhibition (scientific and educational) on April 15-17.

The fifty-second meeting of French Scientific Societies will be held at the Sorbonne on April 14-18.

The Mathematical Association.

REPORT OF THE COUNCIL FOR 1913.

DURING the year 1913, 76 new members have been elected, and the number of members now on the Roll is 744. Of these 8 are honorary members, 31 are life members by composition, 45 are life members under the old rule, and 660 are ordinary members. The number of associates is about 200.

The Council regret to have to record the death of Sir Robert S. Ball, F.R.S., Lowndean Professor of Astronomy and Geometry in the University of Cambridge, who was President of the Mathematical Association for the years 1899 and 1900; and also of Dr. Richard Wormell, who was one of the founders, and a most active member, of the Association (then called the *Association for the Improvement of Geometrical Teaching*) in the year 1871, and has remained a member ever since.

It is proposed to form another Committee of the Mathematical Association, to act as the National Sub-Committee of the International Commission on the Teaching of Mathematics, which will bring its work to a close at the Mathematical Congress to be held in Stockholm in 1916. This Sub-Committee will take charge of the preparation of Reports in answer to the questions circulated among the different countries, and will appoint delegates to attend the Preliminary Congress which will meet in Paris in 1914 and in Munich in 1915.

The Teaching Committees have held many meetings during the year, and those committees which represent the Public Schools and the Other Secondary Schools have prepared reports, which have been published in the *Mathematical Gazette*. The Committees have now been at work for nearly two years, and the Honorary Secretary of the General Committee will lay before the members of the Association an account of what has been done.

In accordance with the original regulations, which were only provisional, and with the experience of their working which has now been obtained, a permanent scheme for the future appointment of the Teaching Committee has been drawn up. The object of one suggestion is the maintenance of some form of continuity in the *personnel* of each of the Committees. A change in the constitution of the Girls' School Committee is proposed, because only a minority of that Committee, as elected in 1912, consisted of those engaged in school-teaching. A copy of the new scheme has been sent to all the members and associates of the Association, and the scheme will be laid before the annual meeting.

The Council again desire to express their very hearty appreciation of Mr. Greenstreet's work as editor of the *Mathematical*

Gazette, and to thank the authorities of King's College and the London Day Training College for their kindness in affording accommodation for the numerous meetings which have been held in connection with the work of the Association.

THE LONDON BRANCH.

THIS Branch has met six times during the year, and all the meetings have been well attended.

At the first meeting, Dr. A. N. Whitehead was elected President for the year, and delivered an address, which dealt with the objects to be kept in view in teaching Mathematics. This was followed by a paper by Mr. J. Katz on "The Use of the History of Mathematics in teaching Elementary Mathematics."

In the summer term the Branch met at the Haberdashers' Aske's Girls' School, Acton, and witnessed a dramatic performance by the pupils, founded on "Flatland."

In November the annual social evening was held at the Regent Street Polytechnic. There was an exceedingly good musical programme, and an interesting lecture by Mr. Briggs on "The Mechanics of Locomotion." We gathered that many of the ideas usually held by engineers and mathematicians on this subject are all wrong, but time did not allow Mr. Briggs to develop his views fully.

At other meetings during the year, the following papers have been read:

(i) "The Parallel Axiom," * by Rev. W. B. Frankland.
(ii) "The Teaching of Elementary Geometry and Trigonometry," * by Mr. W. J. Dobbs.
(iii) "The Teaching of Easy Calculus to Boys," * by Mr. W. Knowles.
(iv) "On a Diagrammatic Solution of Cubic Equations," by Mr. A. Lodge.
(v) "A School Mathematical Club," by Miss M. J. Griffith.
(vi) "Progressions," by Mr. F. C. Boon.
(vii) "Experiments on Water Jets," by Mr. C. J. L. Wagstaff.

THE NORTH WALES BRANCH.

THE North Wales Branch has held three meetings this year, which have been fairly well attended. They have afforded opportunities for professors and lecturers who examine, and teachers who prepare pupils for examination, to exchange views on this subject. The summer meeting was held in Anglesey, and Mr. G. B. Mathews introduced to the members of the branch in a very interesting paper the somewhat complicated subject of countable sets. At the last meeting Prof. G. H. Bryan spoke about harmonics, and gave some interesting musical illustrations on the piano-player, showing how by an ingenious mechanical arrangement proper emphasis may be obtained on certain notes in different portions of the score.

THE SOUTHAMPTON BRANCH.

THE annual meeting took place in King Edward VI. Grammar School, Southampton, on Feb. 14th, 1913. Professor E. L. Watkin of the Hartley College was elected President and Mr. C. H. Holmes, Hon. Secretary and Treasurer.

* These papers have appeared in the *Gazette*.

Four meetings were held in 1913, with an average attendance of five out of a total membership of nineteen.

On February 14th a discussion was initiated by the Hon. Secretary on the "Validity of the Rotation Proofs in Elementary Geometry." The meeting came to the conclusion that these were perfectly sound for Euclidean space, provided that the following axioms are laid down or borne in mind: (1) Translation and Rotation in the Euclidean plane are quite independent processes; (2) Angles in a plane can be compounded by the Arithmetical Law of Addition.

On May 16th the President exhibited a large number of stereoscopic slides illustrating the "Propositions of Solid Geometry."

On October 31st the Hon. Secretary gave an account of his experiences at the Edinburgh Mathematical Colloquium in August.

On November 28th Professor Maxwell read a paper entitled, "The Infinite in Mathematics." The Professor confined himself chiefly to an exposition and discussion of Dedekind's definition of a system of infinite elements.

THE CONSTITUTION OF THE TEACHING COMMITTEES OF THE ASSOCIATION.

I. Special Committees.

SPECIAL Committees for the consideration of problems peculiar to different sections shall be constituted as follows:

1. A Committee of twelve Members of the Association, who are masters teaching in Public Schools, shall be appointed as follows:

(*a*) The three members of the Special Committee who obtained the highest number of votes at the previous election, and are willing to serve, shall remain on that Committee for a further period of two years.

(*b*) Members and Associates of the Association shall be invited to propose members for election. The number proposed by any one individual may not exceed nine.

(*c*) A voting paper containing the names of all those thus proposed, and willing to serve, shall be sent to all Members and Associates of the Association who are masters at Public Schools, each of whom may vote for not more than nine of those proposed.

(*d*) The nine obtaining the highest number of votes, with the three mentioned under (*a*) above, shall constitute the Public Schools Committee. In the case of a tie, the Council shall decide.

(*e*) The Committee shall have power to co-opt not more than three others, who need not necessarily be Members or Associates of the Association at the time of their election, but who shall have no vote until they have joined the Association either as Members or as Associates.

NOTE.—By a "Public School" is meant a school which has a place in the Public Schools Year Book.

2. A Committee of twelve Members of the Association, who are masters teaching in "Other Secondary Schools for Boys," *i.e.* in Secondary Schools which have no place in the Public Schools Year Book, shall be appointed as follows:

(*a*) The two members of the Special Committee who obtained the highest number of votes at the previous election, and are willing to serve, shall remain on that committee for a further period of two years.

(*b*) Members and Associates of the Association shall be invited to propose members for election. The number proposed by any one individual may not exceed ten.

(*c*) A voting paper containing the names of all those thus proposed, and willing to serve, shall be sent to all Members and Associates of the Association who are masters at Other Secondary Schools, each of whom may vote for not more than ten of those proposed.

(*d*) The ten obtaining the highest number of votes, with the two mentioned under (*a*) above, shall constitute the Other Secondary Schools Committee. In the case of a tie, the Council shall decide.

(*e*) The Committee shall have power to co-opt not more than three others, who need not necessarily be Members or Associates of the Association at the time of their election, but who shall have no vote until they have joined the Association either as Members or as Associates.

3. A Girls' Schools Committee shall be appointed, and shall consist of

(i) Twelve women Members of the Association actually engaged in teaching in Girls' Schools.

(ii) The woman secretary of the Association.

(iii) Two representatives of Universities.

(iv) Two representatives of Training Colleges.

The members mentioned in (i) shall be appointed as follows:

(*a*) The three members of the Special Committee who obtained the highest number of votes at the previous election, and are willing to serve, shall remain on that Committee for a further period of two years.

[In the case of the election of 1914 the three referred to above shall be understood to mean the three women Members actually engaged in teaching in Girls' Schools, who obtained the highest number of votes in 1912.]

(*b*) Members and Associates of the Association shall be invited to propose members for election. The number proposed by any one individual may not exceed nine.

(*c*) A voting paper containing the names of all those thus proposed, and willing to serve, shall be sent to all women Members and Associates of the Association, each of whom may vote for not more than nine of those proposed.

(*d*) The nine obtaining the highest number of votes shall be appointed to serve on the Girls' Schools Committee. In the case of a tie, the Council shall decide.

The representatives of Universities and of Training Colleges shall be nominated by the Council.

NOTE.—Any two, or all, of the Special Committees may sit together for the consideration of any matter of common interest.

II. The General Committee.

The General Committee shall consist of

(*a*) 6 representatives of Public Schools.
(*b*) 4 „ „ Other Secondary Schools.
(*c*) 6 „ „ Girls' Schools.
(*d*) 2 „ „ Technical Schools.
(*e*) 2 „ „ Preparatory Schools.
(*f*) 2 „ „ those engaged in training teachers.
(*g*) 4 „ „ Universities.
(*h*) 5, or fewer, other Members nominated by the Council.
(*i*) 5, or fewer, other persons co-opted by the Committee.
(*j*) The President and Honorary Secretaries of the Association.

All members of the General Committee, except the co-opted members, shall be Members of the Association.

1. Representatives under (*a*) shall be the 3 Members of the "Public Schools" Special Committee mentioned in I. 1 (*a*) and the 3 Members of that Committee who receive the highest number of votes at the present election.

2. Representatives under (*b*) shall be the 2 Members of the "Other Secondary" Schools for Boys Special Committee mentioned in I. 2 (*a*) and the 2 Members of that Committee who receive the highest number of votes at the present election.

3. Representatives under (*c*) shall be the 3 Members of the "Girls' Schools Special Committee" mentioned in I. 3 (*a*) and the 3 Members of that Committee who receive the highest number of votes at the present election.

NOTE.—In the case of a tie, or of an uncontested election, the Council shall decide.

4. Representatives under (*d*), (*e*), (*f*), and (*g*) shall be nominated by the Council.

III. All Committees.

1. The Committees shall hold office until their successors are appointed or until an annual general meeting of the Association decide that the Committees shall be dissolved.

2. Elections shall be held at intervals of two years, the elections to take place as soon as possible after the annual general meeting of the Association.

3. Any vacancy on a committee shall be filled by co-option.

4. Any elected member of a committee shall be eligible for re-election.

5. At the time of any election, all co-opted and nominated members of any committee shall cease to be members of that committee, but shall be eligible for reappointment.

6. The Council shall appoint conveners for the first meetings of the General and Special Committees as constituted above; at these meetings the committees shall elect their own officers.

IV. Reports.

1. The General Committee may draw up its own reports; it may appoint sub-committees, and may submit questions to one or more of the

Special Committees. Reports of the General Committee shall be recommended to the Council for publication. Unanimous reports, if published, shall be published as Reports of the Committee. Other reports shall be signed by those members of the Committee who approve of their contents, and, if published, shall be published as the reports of the signatories only.

2. A Special Committee may draw up reports on questions relating to matters within its province. Such reports shall be signed by those members of the Special Committee who approve of their contents, and shall be then forwarded to the General Committee. The General Committee may (*a*) recommend such reports to the Council for publication as reports of one or more Special Committees, or (*b*) adopt them, with the consent of the Special Committee or Committees, and recommend them to the Council for publication as reports of the General Committee, or (*c*) refer them back to the Special Committee or Committees.

3. Reports recommended to the Council for publication (as above) shall be published by the Council unless the Council considers that it is inexpedient to do so for financial reasons.

REPORT OF THE GENERAL TEACHING COMMITTEE, 1912-1913.

1. During the two years of its existence the General Committee has met eight times. Mr. A. Siddons was elected Chairman, and Dr. T. P. Nunn, Secretary. On Dr. Nunn's resignation in October, 1912, Mr. P. Abbott was appointed Secretary.

2. The following were appointed to the Committee by the Council of the Association: Mr. C. S. Jackson, Rev. E. W. Barnes, Mr. H. T. Gerrans, Professor Barrell, Messrs. P. Abbott, J. S. Norman, E. Kitchener, W. J. Greenstreet, and D. Muir.

The following were co-opted by the Committee: Mr. G. St. L. Carson, Mr. G. H. Hardy, and Dr. T. J. I. A. Bromwich.

3. Recommendations to Public Examining Bodies:

(1) *Oxford and Cambridge Schools Examination Board.* This Board communicated with the Committee respecting proposed changes in the mathematics syllabus for the Higher Certificate Examination. Several recommendations were sent to the Board suggesting, *inter alia*, a revision of the algebra syllabus and the retention of alternative papers.

(2) *Cambridge Local Examinations.* The Committee were invited by the Syndicate to offer suggestions with respect to the mathematical syllabuses for the Local Examinations, which were revised for the examinations of 1914. Detailed suggestions, covering all the syllabuses, were submitted, and the Committee is glad to be able to report that a large number of these were accepted by the Syndicate and incorporated in the new regulations.

(3) *Oxford Local Examinations.* Suggestions similar to those drawn up in connection with the Cambridge Locals were also submitted to the Oxford Delegacy. These are still under consideration by the Delegacy.

(4) *London University Matriculation.* The syllabuses in Elementary and Higher Mathematics for the London University Matriculation

were carefully considered by the Committee, and a series of recommendations were made. The Committee regrets, however, that the Senate is not prepared, at present, to make any alterations in these syllabuses.

4. *Reports of Special Committees.*

(1) *Public Schools' Committee.* A report was drawn up by this Committee on "A general mathematical syllabus for non-specialists." This was approved for publication by the General Committee and was published in the *Mathematical Gazette* for July, 1913.

(2) *Other Secondary Schools' Committee.* A report on the Mathematical Teaching in Secondary Schools has been drawn up by this Committee. It has been approved for publication and will shortly be published in the *Gazette.*

(3) *Girls' Schools Committee.* An investigation has been conducted by this Committee into the Teaching of Mathematics to girls in Secondary Schools. A report is being prepared and will be ready shortly.

5. *Future constitution of the Committee.* This matter has been considered by the Committee and certain recommendations, which were the outcome of experience in the working of the Committee, were submitted to the Council.

P. ABBOTT,
Hon. Secretary.

SUMMMER MEETING OF THE LONDON BRANCH.

FLATLAND.

THIS Branch met at the Haberdashers' Aske's Girls' School, Acton, on Friday, June 13th, 1913. The programme for the meeting was of a novel character; instead of the usual papers there was a dramatic performance by the pupils of the school, based on Dr. Abbott's *Flatland.*

The entertainment took place in the School Hall, the audience being seated in the gallery, thus looking down on the performers. In the first place, one of the pupils gave a short description of Flatland, explaining the following points:

The audience were to imagine that they were looking down on an infinite plane, the inhabitants of which were aware of only two dimensions. An enormous pentagon, with gaps in its sides, drawn on the floor, represented a building. Flatlanders being unable to move out of their own plane, required nothing more than lines for walls, height having no meaning for them; the gaps in its sides were the doorways. These inhabitants were shaped like geometrical figures, and their social position was determined by the number of their sides. Thus, the many-sided figures constituted the nobility, whilst the artisans were triangles. Also, the more nearly a figure approached the regular shape, the higher the standing; thus an equilateral triangle was considered above an isosceles triangle, a square above a parallelogram, while a circle was the highest of all. By means of careful education the son of a quadrilateral, say, might in time become a pentagon, or even in some cases a son might be born possessing more sides than his father. The women, however, of all grades were merely straight lines, and so were invisible, when viewed from before or behind; in consequence they might cause considerable damage by running into those coming in the opposite direction, and, to avoid this, they were compelled by law to make a continual humming sound whenever they were in public places, so that they could be heard even though they might not be seen. As a further protection, women had to enter and leave all buildings by special doors.

In Scene I. the inhabitants of Flatland were introduced to the audience by the narrator. They were represented by cardboard models of circles, pentagons, etc., carried horizontally on the girls' heads, thus showing the flat surfaces to the spectators seated in the gallery above.

The following characters marched round the hall and were introduced by name:

DRAMATIS PERSONAE.

THE PRIEST'S FAMILY—	Nine Points Circle,	*Priest.*
	Asymptote,	*Wife.*
	Radial,	*His son.*
THE NOBLEMAN'S FAMILY—	My Lord Manysides.	
	Perpendicular,	*Wife.*
	Aequalis, Planus,	*Sons.*
	Directrix, Tangenta,	*Daughters.*
THE MERCHANT'S FAMILY—	Mr. Protractor.	
	Bisecta,	*Wife.*
	Absurdus, Given, Cyclus,	*Sons.*
	Alterna, Corresponda,	*Twin daughters.*
THE ARTISAN'S FAMILY—	Mr. Acutus.	
	Similar,	*Wife.*
	Policeman *X*, Scalene,	*Sons.*
	Axis, Diameter,	*Daughters.*
PRIEST—	His Reverence the Circumcircle.	
NOBLEMAN—	Sir Multilatus.	
SCHOOLMASTER—	Mr. Ratio.	
	Mr. Rectangulus,	*A heretic.*
	Mr. Slantsides.	
	Base, Altitude,	*His daughters.*

SCENE II.—AT PLAY.

Here the children "dance" and play "patterns," that is, they arrange themselves so as to form the well-known figures of geometrical propositions. These games would appear very dull and uninteresting to our children, who by virtue of their third dimension are enabled to play so many games with a sphere!

In the course of this scene one child relates a queer dream he had, in which he visited a world of one dimension. Here all the inhabitants were lines and moved in lines. They could walk backwards and forwards along their own lines without turning, but could never pass each other; their only means of communication, except with their immediate neighbours, was by shouting. The dreamer relates that these one-dimensional beings could not see him, although he was close beside them, but they could hear him. He says he insulted their king, to whom he talked, by taking him for a woman, seeing that he was only a line. These creatures were much incensed at the folly of the idea of there being a "right" or "left"—"before" and "behind" they could understand, but "right" and "left" had no meaning for them.

Scene III. opens with the master awaiting the arrival of his pupils in the schoolroom. When these arrive they wish their master good morning, and

greet him with a shake of the head—a bow or courtesy being, of course, impossible to a Flatlander. Lessons begin: the essential thing for a child of Flatland to learn is to recognise the size of angles by touch, so that he may leave school well equipped for later life. For, when two Flatlanders meet, they feel one another's angles, and thus recognise the grade of the person they have met; a mistake in this might lead to great social trouble. Considerable time, therefore, is given to this subject in the school curriculum, and some of the children become very clever at it, being able to tell an angle to within a degree by merely feeling it.

The next subject for the day is so-called Algebra—in this they say their tables and then go on to squares, cubes, and symbols. Then follows a lesson on dimensions. The schoolmaster shows with models that a line is formed by the movement of a point, and an area by the movement of a line. One of the pupils, evidently a thinker, besides being the dreamer of Scene II., tries to connect this lesson with the last, and says: "If you can both square and cube x, why should you not be able to get another figure beyond a square when you multiply the length of the line by itself again? And what does an area form when it is moved?" The schoolmaster thinks these are foolish questions, and shows that an area cannot be moved so as to generate anything else than an area. Just then a voice is heard to say "Move it up." The Flatland characters cannot see who is talking, for the voice proceeds from a sphere which is above the Flatland plane, and therefore invisible to them. The master ponders over this so deeply that he does not realise that all his pupils have run away, and the scene closes with the schoolmaster trying, in a very puzzled way, to find out what is moving his model "up," a word hitherto unknown to Flatlanders.

In Scene IV. Lord Manysides and Sir Multilatus meet in the street and recognise each other in the way indicated in the previous scene. While talking, they encounter Mr. Protractor, who, though only a merchant, is intelligent beyond his sides, and has become a friend of Sir Multilatus. He introduces Mr. Protractor to Lord Manysides, the method of introduction being peculiar; first, Lord Manysides feels all Mr. Protractor's sides and angles, so that he will be able to recognise him by touch when next they meet, and then the operation is repeated by Mr. Protractor. After some conversation, Lord Manysides invites Mr. Protractor and his wife to dinner.

Scene V.—House of Lord Manysides.

Lord Manysides, on returning from his walk, tells his wife that he has asked Mr. and Mrs. Protractor to dinner. She is so angry at his having asked a mere merchant and his wife to dinner that she runs at her husband as hard as she can, and kills him by piercing his polygon with her rod. She is overcome by grief and remorse, and makes no resistance when Policeman X comes to arrest her.

Scene VI.—In the Prison.

As stated in the Prologue, walls are unnecessary either in houses or prisons, the only essential thing for the latter being that the door-spaces should be securely filled up.

There are several prisoners present when Lady Manysides is brought into the prison, and these are discussing the various causes for their imprisonment. One says he is there for fraud—being an isosceles triangle, he applied for a post open only to a rhombus, and obtained it because he and his son walked with their bases in contact.

Another is our old friend the dreamer, who is charged with heresy because he would persist in discussing the possibility of a third dimension.

SCENE VII.—IN THE LAW COURT.

The trial of the heretic is proceeding, and everybody is pointing out the absurdity of the idea that there could be a third dimension, when a new witness suddenly appears, entering where there is no door, much to the surprise of everybody in court. However, he looks like a circle, so all make way for him, supposing him to be of the nobility; but they notice this peculiarity about him, that his size is continually changing, and sometimes he disappears altogether. The fact is that he is a sphere, and he thinks to convince them of his third dimension by bobbing up and down, and by jumping over their line wall. But the point is quite lost on the Flatlanders, who cannot conceive such an idea. He tries hard to explain himself to those present, and lets them feel him all over, but they are all quite mystified, except the dreamer, who at last sees some glimmer of light on his doubts.

The court breaks up; the prisoner is released and goes away with the sphere to learn more of this strange phenomenon.

With this the performance ended amid great applause.

After the performance there was an interval for refreshments, which the Headmistress, Miss Gilliland, had very hospitably provided. A discussion followed—partly on the subject-matter of the performance, partly on the educational aspect of such performances. One or two speakers threw out suggestions as to the possibility of there being a fourth dimension, unknown to us and as seemingly impossible as a third was to the inhabitants of Flatland.

Miss Brown and Miss Griffiths (both teachers at the school) stated how much the interest of the girls had been roused by the preparations involved. The discussions on the nature of Flatland, the making of the cardboard models, the search for inconsistencies in the representation of Flatland, the enquiries about the meanings of the names of the characters, all these were of educational value, not only to the performers, but also to other girls in the school.

Some of the audience, however, doubted whether the mathematical advantages reaped were a fair equivalent to the time and trouble devoted to the preparation of the play by pupils and teachers.

The proceedings terminated with a hearty vote of thanks to Miss Gilliland and the staff of the school.

THE London Branch met at the London Day Training College on Saturday, Oct. 25th, at 3 p.m. The President, Dr. Whitehead, was in the Chair, and there was a good attendance.

Mr. Alfred Lodge (Charterhouse) gave a brief description of a novel diagrammatic solution of cubic equations.

Mr. Knowles (Borough Polytechnic Institute) read a paper on "The Teaching of Easy Calculus to Boys."

MATHEMATICS IN SECONDARY SCHOOLS.

A Report drawn up by the "Other Secondary Schools" Special Committee and approved by the General Teaching Committee of the Mathematical Association.

THE Committee are of opinion that the Mathematical teaching in Secondary Schools is too much influenced by the necessity for preparing the boys for certain external examinations. They desire to call attention to the fact that the great majority of the boys leave school before they reach the age of 17, and that much mathematical matter has to be omitted or unduly postponed which would be useful to them at school in Science and other subjects, as well

as in after life. It is hoped that examining bodies will make arrangements, by setting alternative papers or otherwise, so that in some Secondary Schools at least the mathematical teaching may follow the lines indicated below.

1. *General Purpose.*—It is desirable that the average pupil should study mainly those parts of mathematics which develop his powers of thinking, and are of practical service as he proceeds in his school course.

2. *Rigour.*—While rapid progress towards the use of mathematical tools is desirable, at the same time it is indispensable that every mathematical statement should be justified or proved in a way suitable to the stage reached by the student, and the mathematical course should include some training in rigorous deduction from certain explicit assumptions. It is desirable that there should be considerable latitude in selecting the assumptions; these, however, should be clearly stated in the teaching.

3. *Algebra.*—It is possible to economise much time by omitting as non-essential certain parts of this subject which have usually occupied too much time. In the hands of properly qualified mathematical teachers much of the manipulation can safely and effectively be taught incidentally.

4. *Trigonometry.*—It is desirable that teachers should be allowed to treat Geometry, Trigonometry, Coordinate Geometry and Mensuration as one subject. Algebraic methods and Trigonometrical functions may be introduced, and used, early in the Geometrical course.

5. *Calculus.*—It is desirable that boys of ordinary ability who attend a Secondary School to the age of 16 should not leave without some introduction to the principles on which the Calculus is based.

6. *Mechanics.*—Though this subject is based on experiment, it is particularly well adapted to a mathematical treatment, and should form part of the Mathematical course.

7. *Solid Geometry.*—There is too great a tendency to limit school mathematics to two dimensions. It is desirable that a simplified study of Solid Geometry should be commenced early—perhaps in connection with Geography, Astronomy, and Carpentry. The Committee feel that this subject has been unjustifiably crowded out by examinations. In connection with this, the Committee desire to call attention to Section 7 of the Report of the Public Schools Special Committee:*

P. Abbott, A. Radway Allen, R. Wyke Bayliss, J. V. H. Coates, E. Fenwick, T. J. Garstang, W. Lattimer, C. M'Leod, W. E. Paterson, H. C. Playne, W. J. Dobbs, *Hon. Sec.*, C. J. L. Wagstaff, *Chairman.*

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* The section reads as follows:

SOLID GEOMETRY.

The course should include some simple Solid Geometry, which might be introduced incidentally during the course of Plane Geometry. The main object of including this work is, that the power of thinking in space should be cultivated throughout. The following suggestions indicate some of the ways in which this may be carried out:

(1) Riders on congruent triangles may sometimes deal with triangles not in the same plane.
(2) The theorem of Pythagoras may be applied to figures in three dimensions, *e.g.* to finding the height of a cone, of which the slant height and the radius of the base have been measured.
(3) In dealing with some of the properties of the circle, the corresponding properties of the sphere may be discussed.
(4) Some work may be done on the plan and elevation of simple objects. The purpose of this work is not so much to teach a boy how to draw a plan and elevation of a given solid as to enable him to visualise the solid whose plan and elevation are given.
(5) Problems in Elementary Trigonometry should involve observations in more than one plane.
(6) Examples may be given on the angle between two planes or between a line and a plane. The elementary solids provide material for such exercises.

THE TEACHING OF CALCULUS.

QUESTIONS PROPOSED BY THE SUB-COMMISSION A of the International Commission on Mathematical Teaching, with regard to the position now occupied by the Elements of Differential and Integral Calculus in the programmes of Public and Secondary Schools.

Note 1. The object of the Central Committee in formulating these enquiries has been solely that of acquiring information. The Committee does not itself take up any definite standpoint in the matter as to how far the teaching of the subject in the schools is desirable.

2. By the term "Public and Secondary Schools" is to be understood those Day and Boarding Schools which correspond to the French Lycées and the German Gymnasia and Real-Gymnasia. Information is, however, also desired, whenever possible, with regard to what is being done in the Teachers' Training Colleges. The particular type or types of school in the district considered should always be mentioned, and it should be stated whether the Calculus is part of the official curriculum, or included, or not, at the option of the individual teacher. The percentage of schools in which the Calculus is taught should also be given. It should also be mentioned whether all the pupils are taught the subject, or only some of the more advanced ones.

I. *How much of the Differential and Integral Calculus is taught in the schools of the country under observation?*

In particular:

(*a*) Is the Differential Calculus applied to functions of a single variable, or are functions of several variables also treated?

(*b*) To what specific functions is the Differential Calculus applied?

(*c*) Is the Integral Calculus studied? If so, within what limits?

(*d*) Is Taylor's Theorem discussed?

(*e*) Are simple Differential Equations solved? If so, what?

II. *How far is the treatment of the subject rigid, both as to the mode in which the fundamental concepts are introduced, and as to the demonstrations employed?*

(*a*) Is it considered sufficient to introduce the notions of the Differential Calculus geometrically, without expressly using the idea of a limit, or is this idea explicitly employed? In the latter case, is there an attempt at a rigid presentation of the subject, or are theorems like $\mathrm{Lt}\,\frac{1}{a}=\frac{1}{\mathrm{Lt}\,a}$ taken for granted?

(*b*) Are differentials used? If so, is the Differential Calculus employed as a sort of calculus of approximations, or are infinitely small quantities treated as if they were small quantities which really exist?

(*c*) In Taylor's Theorem is the remainder considered, or not?

(*d*) Is attention called to the fact that there are non-differentiable functions?

(*e*) Is the idea of an irrational number logically and systematically introduced, or is it considered sufficient to speak incidentally of irrational numbers, for instance, in the extraction of square roots?

III. *How is the pupil introduced to the ideas of the Differential and Integral Calculus?*

(*a*) Does he receive a preliminary training in the lower classes of the school, based on the study of appropriate simple functions and their graphs, so that

the matter appears to rise naturally out of the subjects already studied, and not to constitute a supplementary course ?

(*b*) Is Leibniz's Notation employed ? If not, what symbols are used for the differential coefficient and integral ?

(*c*) Which is considered first, the Differential or the Integral Calculus, or are they taught simultaneously ?

(*d*) Is the integral introduced as the limit of a summation (definite integral), or as primitive function (inverse differential coefficient) ? If in both senses, in what order and in what connection with one another are the two points of view considered ?

(*e*) Is a textbook used ? If so, the exact title, publisher, and edition should be quoted.

IV. *What applications of the Differential and Integral Calculus are considered* ?

What questions of analysis (Higher Algebra and Trigonometry), geometry or physics involving the idea of a limit, otherwise wholly or partially present in the programmes of the schools, are utilised to illustrate and explain the Differential and Integral Calculus, so that there may be an economy in the treatment of the subjects studied ?

In particular :

(*a*) Is the Calculus applied to the theory of maxima and minima ?

(*b*) When Taylor's Theorem is considered, what are the functions whose developments in power series are obtained by means of it ?

(*c*) In the cases where the remainder form of Taylor's Theorem is discussed, are power series used for purposes of interpolation, extrapolation and the calculation of errors ?

(*d*) When the Integral Calculus is taught, is it applied to the calculation of areas (in the cases, for instance, of the parabola and ellipse), and of volumes ?

(*e*) In connection with what fundamental concepts of Mechanics (velocity, acceleration, work, moment of inertia, etc.) is use made of the Differential and Integral Calculus ?

(*f*) The corresponding questions for Physics, and in particular for Optics (curves, envelopes, etc.), and for Electrodynamics (lines of force, etc.), should be answered.

V. *Has the introduction of the Differential and Integral Calculus been at the expense of other branches of study* ? *If so, of which* ?

VI. *What has been the result of the recent introduction of the Differential and Integral Calculus into the school programmes* ? *Is the introduction felt to have been an inevitable advance* ? *How far has it found support, or the contrary* ? *In particular, what is the attitude of mathematicians and physicists towards the innovation* ?

Should any other details of interest concerning the teaching of the Differential and Integral Calculus have come to the knowledge of the observer, it is requested that they may be chronicled at this stage of the report.

A list should also be made of the passages in the reports published by the sub-commission in the country in question which relate to the teaching of the Differential and Integral Calculus.

The Report is to be presented in April, 1914, at Paris.

TEACHING OF CALCULUS IN PUBLIC AND SECONDARY SCHOOLS IN THE UNITED KINGDOM.

Public Schools—England.

"Public Schools" correspond broadly to German Gymnasia.

The boys studying calculus in these schools are of three classes:

(1) Boys aged 17 to 19, of some mathematical ability, who will in due time compete for University Scholarships.

(2) Boys of 17 to 19, preparing for the Army entrance examinations, or destined for the profession of engineering.

(3) Boys aged 16 to 18, of merely average ability, who take the calculus as a regular item of school mathematics.

Class (1) may be assumed to be in the highest mathematical class of the school. In this class, there are usually several small groups of boys of various degrees of advancement; it is commonly impossible to make much use of class teaching, except in special subjects; as a rule, the boys work away at their textbooks and exercises, and come up to the master's desk for elucidations in their turn. Under these circumstances the lines of their work are determined by the textbook they are using. I have therefore thought it sufficient to examine the textbooks most commonly in use, and to answer the questions proposed in the light of this examination and of my own personal impressions.

The number of boys of this class studying calculus at a given time in one school is small. There is nothing new in the fact of such boys studying calculus; it was not unusual in my own schooldays, 20 to 25 years ago, but no doubt is much more general now.

Class (2) have come into being in the last 15 years or so. On the one hand, calculus of a practical description has within this period been included in the Army entrance examinations. On the other, the growth of engineering courses at the Universities, together with more general causes, has drawn the attention of more school boys to the engineering profession as a possible opening; a demand has therefore arisen for a suitable preparatory course at schools. Boys in Class (2) are usually of less outstanding mathematical ability than those in Class (1). Classes (1) and (2) work together in some schools; but more usually there is a special Army class. I cannot make any general statement as to the provision most usually made for the engineering boys; circumstances vary very much.

It may be said that the instruction given to boys of Classes (1) and (2) tends to differ in this respect; that for Class (1) the teaching has more mathematical rigour and in Class (2) has a more practical flavour.

Class (3) represents a more recent movement. "Calculus for the average boy" is the keynote of the modern movement in mathematical teaching. The pressure of external examinations urges in this direction. In a few schools a preliminary course in calculus now forms an organic part of the curriculum for the middle classes; in many schools there is a tentative movement in the same direction. The work done under this head is concerned with the gradient of a graph and rate of increase of a function, with applications to maxima and minima, small increments, velocity and acceleration. The inverse problem of integration with the obvious applications is examined in some schools. The functions dealt with are usually simple integral powers of x.

It may be doubted whether the movement in question has yet produced its full effect, and probably a report on the subject at the present moment would be premature.

Textbooks. An enquiry as to textbooks in use was addressed to the 113 public schools. Replies were received from 94 schools.

It appears that two or three different books on calculus are in use in most schools. It is a very common practice to use a book such as Mercer or Gibson

with beginners, and to pass on to such a book as Edwards at a later stage. The books most commonly used in public schools are the following:

J. Edwards, *Differential Calculus for Beginners* (Macmillan). 1st Ed. 1892. 262 pages.

J. Edwards, *Integral Calculus for Beginners* (Macmillan). 1st Ed. 1894. 308 pages.

J. Edwards, *Treatise on the Differential Calculus* (a more advanced work) (Macmillan). 1st Ed. 1886.

J. W. Mercer, *Calculus for Beginners* (Cam. Univ. Press) 1st Ed. 1910. 440 pages.

* G. A. Gibson, *Introduction to the Calculus* (Macmillan). 1st Ed. 1904. 225 pages.

H. Lamb, *Infinitesimal Calculus* (Cam. Univ. Press). 1st Ed. 1897. 616 pages.

A. Lodge, *Differential Calculus for Beginners* (Bell). 1st Ed. 1902. 4th Ed. 1913. 299 pages.

A. Lodge, *Integral Calculus for Beginners* (Bell). 1st Ed. 1905. 2nd Ed. 1912. 203 pages.

B. Williamson, *Elementary Treatise on Differential Calculus* (Longmans, Green & Co.). 6th Ed. 1887. 468 pages.

B. Williamson, *Elementary Treatise on Integral Calculus.* 6th Ed. 1891. 463 pages.

A. E. Love, *Elements of Differential and Integral Calculus* (Cam. Univ. Press). 1st Ed. 1909. 207 pages.

W. M. Baker, *Calculus for Beginners* (Bell). 1st Ed. 1912. 166 pages.

From the returns obtained it appears that an examination of the books by Edwards, Mercer and Gibson should give a fair idea of the teaching.

Edwards' books are very popular, and have had a long reign. They date from a time when rigour of treatment was little appreciated in England, even at Cambridge. I have never used these books myself, but I attribute their popularity to their usefulness for examination purposes, and to their good collections of exercises. They do not appear to contemplate the early teaching of the calculus to boys of average ability.

The growing prevalence of such early teaching is indicated by the success of the books by Mercer and Gibson; and the existence of such books has, no doubt, a powerful reaction on the movement. A distinguishing feature of both books is the sound and patient exposition of the fundamental points. In both books the treatment of integral calculus is interlaced with that of differential calculus, the former being attacked as soon as the latter has been developed to a certain point. Books of this type are found very suitable for introducing the beginner to the subject. A sufficient idea of the methods adopted may be gained by an inspection of the tables of contents, which are reproduced in the Appendix to this paper.

I. *How much of the Differential and Integral Calculus is taught in Public Schools?*

This depends entirely on the proficiency of the individual pupils.

In particular:

(*a*) Functions of more than one variable are treated by Edwards, but not by Mercer or Gibson.

(*b*) The Differential Calculus is applied to ordinary algebraic functions, and to exponential, trigonometrical and logarithmic functions. In a minimum course designed for the average student the tendency is to use only x^n.

(*c*) The Integral Calculus is generally studied by those who have made

* Prof. Gibson has also written a large book called *An Elementary Treatise on the Calculus.* The references below are always to the *Introduction to the Calculus.*

some progress with Differential Calculus. The limits depend on the ability of the student.

(*d*) Taylor's Theorem appears at an early stage in Edwards. Mercer does not give it, but discusses the approximate relation

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x).$$

Gibson does not give Taylor's Theorem; but he obtains developments of $\sin x$, $\cos x$, $\tan^{-1}x$, $\log(1+x)$, $\log\frac{1+x}{1-x}$ as series to n terms approximately representing the functions, giving in each case an upper limit for the error. He also sketches the exponential series.

(*e*) Edwards deals with Differential Equations at a late stage. (Equations of first and second order, and linear equations with constant coefficients.) Gibson does not give separate chapters to the subject, but solves incidentally a few differential equations connected with kinematics. Mercer points out that $\frac{dy}{dx}=2x+3$ gives a family of curves: he also includes a set of examples on differential equations, in which the methods of solving some of the common types are indicated, most of the work being left to the student.

II. *How far is the treatment of the subject rigorous, both as to the mode in which fundamental concepts are introduced, and as to the demonstrations employed?*

Twenty years ago, when I was up at Cambridge, the teaching given to ordinary students at that University was not conducted on rigorous lines. A movement towards rigour of mathematical treatment was then just beginning to make itself felt, and since that date has, I believe, gone to considerable lengths. Many of the best prepared students turned out at that date and earlier are unfamiliar with the modern standard of rigour, and presumably do not employ it in the teaching they now impart to others. Belonging to the non-rigorous period myself, I do not feel myself entirely competent to criticise the standard of rigour adopted in various textbooks. Edwards does not, I imagine, satisfy modern standards. As far as I am able to judge, Mercer and Gibson have spared no pains to convey accurate ideas within the scope of their work. But books of this type do not pretend to be exhaustive in their discussion of fundamentals. Their ideal appears to be this—state nothing but the truth, but do not necessarily state the whole truth.

(*a*) The idea of a limit is, as a rule, employed explicitly; it is not considered sufficient to introduce the notions of the Differential Calculus geometrically. On the other hand, such theorems as $\mathrm{Lt}\,\frac{1}{a}=\frac{1}{\mathrm{Lt}\,a}$ are generally taken for granted in teaching.

(*b*) Differentials are not used very prominently. I imagine that the impression formed in the mind of most school boys is that infinitely small quantities are small quantities which really exist. But this notion could not arise from a careful perusal of Mercer or Gibson.

Lodge bases his whole work on differentials. "In applications of the Calculus to mensuration, physics, mechanics, etc., the notion of differentials is fundamental, and, in particular, integration as a process of summation is based on that notion. Moreover, the simplicity of a differential lies in the fact that its 'dimensions' are the same as those of the integral quantity from which they spring, whereas differential coefficients are new quantities whose dimensions depend on those of *two* or more of the variables under consideration. The notion of differentials is some thousands of years older than that of differential coefficients, so that, historically, the order of precedence is justified."

Lodge claims that the method stimulates the intuitive perceptions of students to a remarkable degree. He has used it now for 30 years.

(*c*) The remainder in Taylor's Theorem is not considered on the first introduction of the series by Edwards, but is discussed at a later stage. Lamb avoids Taylor's Theorem till a late stage, at which the theorem can be discussed properly. Mercer and Gibson do not give the theorem.

(*d*) Lamb points out that non-differentiable functions exist, but will not be met with in his book. I have found no reference to the matter in Edwards, Mercer or Gibson.

(*e*) Is the idea of an irrational number logically and systematically introduced ?

As a rule, no. It is generally held that a satisfactory discussion of irrationals is work of university type. But, of course, there are teachers who hold and act upon the view that a strict treatment should enter from the start.

III. *How is the pupil introduced to the ideas of the Differential and Integral Calculus* ?

(*a*) The graphing of simple functions is a normal feature in the work of lower classes. This is not perhaps generally undertaken with an eye to the subsequent study of the Calculus; at the same time, the pupil is bound to be more or less familiar with the notion of the graph of a function before he attacks the Calculus.

(*b*) The notation generally employed is that of Leibniz. There is a tendency, however, to postpone the difficulties of this notation by using at first such notations as $Df(x)$; $D_x f(x)$.

(*c*) The older books contained separate volumes for Differential and Integral Calculus. The newer books tend to interlace the two branches to some extent.

Edwards begins his Integral Calculus with the determination of area by limit of summation; and evaluates by the same method the integrals of various functions. He deduces that integration is the inverse of differentiation.

Mercer begins by proposing the problem, given $\frac{dy}{dx}$, find y. Then follow applications to families of curves, and to kinematics. The next chapter opens with a geometrical and arithmetical discussion of area regarded as a summation. This is followed by the introduction of the equation $\frac{d}{dx}\text{area} = y = f(x)$; hence definite integral. Then various exercises; and ultimately the notation $\int$.

Gibson's treatment differs from Mercer's in that (1) $\frac{d}{dx}(\text{area}) = y$, and the definite integral precedes the idea of summation; (2) the notation $\int$ is introduced at the outset.

IV. *What applications of the Differential and Integral Calculus are considered* ?

It may be stated generally that the economies that might result from an application of the Calculus are not fully realised in practice. This is due to the custom of reading analytical geometry and mechanics before the calculus. For instance, the calculus is not usually applied (1) to ascertain the distance traversed by a point whose velocity varies; (2) to find the equation of the tangent to a plane curve.

(*a*) The Calculus is applied to the theory of maxima and minima, but the maxima and minima of a quadratic function are generally investigated algebraically at an earlier stage.

(*b*) Functions such as $(1+x)^n$, $\sin x$, $\cos x$, $\tan^{-1} x$, a^x, $\log(1+x)$ are usually developed in power series before Taylor's Theorem is reached.

(*c*) In cases where the remainder form of Taylor's Theorem is discussed, are power series used for purposes of interpolation, extrapolation and the calculation of errors? Work of this kind is not usually undertaken.

(*d*) The Integral Calculus, when taught, is always applied to calculation of areas and of volumes.

(*e*) The Calculus, when taught, is applied to deal with velocity, acceleration, work (*e.g.* stretching string, expanding gas), fluid pressure on areas, centre of mass, moment of inertia.

(*f*) Branches of physics, other than mechanics, are not usually taught by mathematical masters. I have no precise information as to how far physics masters use the calculus in such subjects as Optics and Electrodynamics. But it may be assumed that the majority of physics masters in public schools concern themselves with the experimental and descriptive rather than the mathematical side of their subject; the proportion of masters who have made an extended study of both physics and mathematics is not great. Now that the new tripos regulations have come into force at Cambridge, the number of such masters may be expected to increase. If the masters teaching physics and mathematics were more closely in touch, the former would be able to make more use of the rudimentary ideas on calculus that the boys have obtained from the latter.

V. *Has the introduction of the Differential and Integral Calculus been at the expense of other branches of study? If so, of which?*

As regards boys of Class (1) (see page 235), the introduction of the calculus is not recent. In so far as there has been increased emphasis laid on the calculus in recent years, the time has probably been subtracted from such topics as trilinear coordinates, geometrical conics or theory of numbers.

As regards boys of Classes (2) and (3), calculus tends to encroach upon algebra, threatening the more manipulative and formal side of this subject. There may have been some economy, too, in arithmetic, especially in the matter of obsolete commercial rules. Less frequently, the formal side of geometry suffers.

VI. *What has been the result of the recent introduction of the Differential and Integral Calculus into the school programmes? Is the introduction felt to have been an inevitable advance? How far has it found support, or the contrary? In particular, what is the attitude of mathematicians and physicists towards the innovation?*

In answering this question in the absence of definite replies from a large number of correspondents, it is difficult to eliminate one's own personal views and aspirations. The subject has been ably discussed by Mr. C. S. Jackson, in a paper entitled, "The Calculus as a School Subject," which is incorporated in Part I. of the *Reports on the Teaching of Mathematics in the United Kingdom*, as presented to the Cambridge Congress in 1912. Mr. Jackson's attitude may be described as sympathetic but critical.

Broadly speaking, the movement has received general support. Perhaps the most powerful stimulus is that of the engineers, as represented by Prof. Perry. The physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigour. The Universities have progressively included more calculus in their examination papers for schools; these papers, together with those set by the Civil Service Commissioners (for admission to the Army and the public service generally), are the most powerful lever that acts on the school curriculum. It will be understood, of course, that there is in England no general curriculum imposed upon schools: schools frame their own curricula, but tend to adapt them to the examination requirements of their pupils.

Whatever opposition there has been to an introduction of the calculus at an early stage has come from those who fear that a diminished emphasis on

the manipulative and formal side of algebra will have a bad effect. The question raised is this: What algebraic equipment constitutes a firm base for a superstructure of Calculus?

This is the only articulate objection that has found voice. But the main obstacle is that most powerful force in educational matter—*vis inertiae*.

I submitted a first draft of this report to the members of the Public School Sub-committee of the Mathematical Association. I have to thank many of these gentlemen for suggestions which I have been glad to incorporate in the final report. It must not be understood, however, that anyone shares with me the responsibility for the statements made above.

Prof. Gibson informs me that my remarks may be taken as generally applicable to the Secondary Schools of Scotland. C. GODFREY.

THE TEACHING OF NUMERICAL TRIGONOMETRY.

(*Concluded.*)

A LITTLE experience of problems in which it is necessary to divide by sine or cosine of an angle will prepare a boy for the definitions of secant, cosecant and cotangent as the reciprocals of the cosine, sine and tangent respectively, though, as a matter of fact, he can get on very well without them. At this stage a boy will probably appreciate the fact that since an acute angle is determined if we know any one of the six trigonometrical ratios, it should be possible, if any one of the ratios is given, to find the others. Problems such as this: "Given $\sin A = \cdot 64$, find $\tan A$" should be done (i) by drawing a figure to scale, *i.e.* constructing the angle A whose sine is ·64, and finding the tangent by measurement; (ii) by drawing a right-angled triangle, hypotenuse 100 units, one side 64 units, and calculating the other side by Pythagoras' theorem; (iii) by taking from the tables the angles whose sine is ·64 and looking up its tangent.

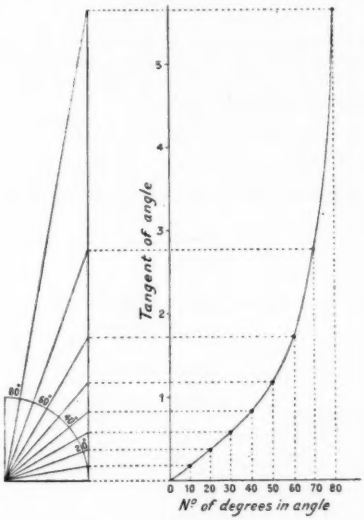
With judicious prodding, he should have no difficulty in discovering such relations as $\tan A = \dfrac{\sin A}{\cos A}$, $\sin^2 A + \cos^2 A = 1$, and these should be verified in special cases.

Sometime or other the graphs of the ratios will come in. That the tangent changes gradually as the angle changes is obvious from the figure, and it is also obvious that equal increments in the angle do not produce equal increments in the tangent. An inspection of tables, of course, tells the same story, but the graph tells it much better, especially to boys who are trained as most boys are nowadays, to "think in graphs." The general form of the graph should be familiar, the great thing being that it is not straight. It is very difficult even with good boys to keep out such howlers as

$$\tan 40^\circ + \tan 30^\circ = \tan 70^\circ, \quad \tan 80^\circ = 2 \tan 40^\circ,$$

and frequent appeals to the non-straightness of the graph should be helpful. Personally, I am inclined to put this graph quite early, when boys are finding tangents of different angles. The graph appeals to them at that stage as a sort of pictorial table of natural tangents, and should be plotted from the figure from which the tangents are found rather than from the tables. The graphs of $\sin x$ and $\cos x$ should also be familiar. It is worth while, I think, as throwing some light on the difference columns and the reason for their absence in certain places, to plot on a large scale portions of these graphs, *e.g.* $\sin x$ from 30° to 31°, $\tan x$ from 45° to 46°, $\tan x$ from 87° to 88°, using the values given in the tables for 30°, 30·1°, 30·2°, etc. The next stage will consist in solving right-angled triangles in cases where the arithmetic is heavier and logarithms are used to lighten it. The actual steps

of the solution are exactly the same as before: the only difference is in the way in which the arithmetical operations are carried out. After a time, but not too soon, the advantages of co-logarithms and so-called "Logarithmic Sines" may be pointed out. These are, in a sense, luxuries which can be dispensed with, though they lighten the work. By the way, I think it was De Morgan who said that if the logarithm of the sine is called the logarithmic sine, the king of the country should be called the royal country.



In these problems on right-angled triangles, you will have plenty of opportunity of insisting on orderly arrangement of work and accuracy in simple addition, and the sooner boys take an interest in "trimming" their logarithms, *i.e.* keeping the columns vertical, the better for their chances of coming safely through a complicated piece of work.

The general triangle can now be solved by splitting into right-angled triangles. In accordance with the general principle previously mentioned, the examples should be easy to start with, *e.g.* $b=10$, $c=12$, $A=53°\,8'$. There is nothing new here, and with the hint that the triangle is to be split into two right-angled triangles, any boy should be able to complete the solution without further guidance. He may draw a perpendicular either from B or C; it is better to take the one from C, as we know for certain that A and B are acute. If BE is drawn perpendicular to AC, it might happen that C was nearly a right angle, in which case the inevitable uncertainty about the length EC might affect the result.

He will want more help with the case in which the three sides are given, *e.g.* suppose $a=13$, $b=14$, $c=15$ (unit 1 cm.). Obviously A and B are both acute, and the perpendicular CF falls within the triangle. Let AF, FB, FC be x, y, z cms. respectively. A can be found when x or z is known, and similarly for B. Any boy will now be able to write down the three equations

$$x^2+z^2=14^2, \quad y^2+z^2=13^2, \quad x+y=15.$$

The elimination of z from the first two gives

$$x^2-y^2=14^2-13^2 \quad \text{or} \quad (x+y)(x-y)=27;$$

and since

$$x+y=15,$$

$$\therefore\ x-y=1{\cdot}8;$$

$$\therefore\ x=8{\cdot}4 \text{ and } y=6{\cdot}6.$$

Hence

$$\cos A=\frac{8{\cdot}4}{14} \quad \text{and} \quad \cos B=\frac{6{\cdot}6}{13}.$$

A good check is obtained by finding z from $14\sin A$ and $13\sin B$. After this the solution of the general triangle with heavier arithmetic will follow. I give an example of the method of setting out the work in the case when two sides and the included angle are given:

$$b=48{\cdot}29, \quad c=36{\cdot}71, \quad A=56^\circ.$$

$$c<b; \quad \therefore\ C<B; \quad \therefore\ C \text{ is acute.}$$

Draw BE perpendicular to AC, and call AD, CD, BD, x, y, z respectively.

		No.	Log.
From $\triangle BDA$,	(i) $z=36{\cdot}71\sin 56^\circ$	36·71	1·5648
	$=30{\cdot}43$	$\sin 56^\circ$	$\bar{1}$·9186
		z	1·4834
	(ii) $x=36{\cdot}71\cos 56^\circ$	36·71	1·5648
	$=20{\cdot}53.$	$\cos 56^\circ$	$\bar{1}$·7476
		x	1·3124
	(iii) $y=48{\cdot}29-x$		
	$=27{\cdot}76.$		
From $\triangle BDC$,	(iv) $\tan C=\frac{z}{y}$;	z	1·4834
		y	1·4434
	$\therefore\ C=47^\circ 38'.$	$\tan C$	0·0400
	(v) $a=y\sec C$	y	1·4434
	$=41{\cdot}19.$	$\sec C$	0·1714
		a	1·6148

Thus **$a=41{\cdot}19$, $C=47^\circ 38'$, $B=76^\circ 22'$.**

As a check, draw AD perpendicular to BC.

Call AD, p.

From $\triangle ADB$,	$p=36{\cdot}71\sin 76^\circ 22'.$	36·71	1·5648
		$\sin 76^\circ 22'$	$\bar{1}$·9875
		p	1·5523
From $\triangle ADC$,	$p=48{\cdot}29\sin 47^\circ 38'.$	48·29	1·6838
		$\sin 47^\circ 38'$	$\bar{1}$·8685
		p	1·5523

The course I have sketched out is well within the scope of all boys, and with no further knowledge of the subject, they have acquired a powerful weapon. Even the weakest boys feel that they have got hold of something which they can use in all sorts of situations, and they use it quite freely where cleverer boys would perhaps use algebraical or geometrical methods. To illustrate what I mean, consider this problem which I generally set sometime or other to boys working at the geometry of the circle: "A circular arc has its chord AB 10 inches long, and its height CD 2 inches long; find the radius of the circle." Let O be the centre and E the other end of the diameter through D. Occasionally one finds a boy who uses $DC.CE=AC^2$.

More often, algebra is used and the problem reduced to the equation $x^2-(x-2)^2=5$, where x'' is the radius. But in a lower class, most boys fall back on trigonometry—they find $\angle ADC$ (its tangent is 2·5), deduce $\angle COA$, and then get the radius from $OA=CA\operatorname{cosec} COA$. Who shall say that this performance is without merit? If a boy can use trigonometry or algebra to help out his geometry so much the better.

In Elementary Statics and Dynamics the trigonometry of the right-angled triangle is all that is required. For instance, the resultant of two forces inclined at any angle is easily found by resolving one along and perpendicular to the line of action of the other, and so reducing the problem to that of finding the resultant of two forces at right angles, and the same method can be extended to the case of any number of forces acting at a point. As I said at the beginning, and I hope have made clear since, there is plenty of scope in this kind of work for patience, orderly arrangement of work and accuracy, and the definiteness of aim and reality of the problems seem to appeal to boys. The variety of problem that can be set is very great, and a boy's wits are constantly exercised in discovering the best way of connecting what he has to find with what is given. Moreover, I do not see that the boy who is going to be a mathematician will suffer by beginning in this way. He will soon cover this preliminary course and have nothing to unlearn later.

J. W. MERCER.

THE CALCULUS AS AN ITEM IN SCHOOL MATHEMATICS.

(*Continued.*)

DIFFERENCES.

THE step in y.

But to return to the differences. Suppose we want the graph of

$$y=16{\cdot}8x-x^2/2.*$$

x	y	Step in y	
0	0		
		15·3	
1	16·3		
		14·3	−1
2	31·6		
		13·3	−1
3	45·9		
		12·3	−1
		11·3	

A bold induction by simple enumeration has been made. The graph when plotted confirms it. Let elementary algebra justify it. A little later the algebra will come first.

Just when to introduce the symbol Δ is of course a question of knowing the class.

*Of course a simpler example would be given to the schoolboy to begin with—perhaps $y=x^2-x$. The example in the text is intended to show how what would be a somewhat tedious and troublesome graph may be rapidly plotted without algebraic transformation. There is no necessity to complete the y column. All plotting may be done by means of Δy.

How the differences give the slope of certain chords will be evident. How far at this stage to allow the question of the tangent to arise is a question of tact.

It will be noticed that the algebra required for these examples is of the simplest character. Such questions as: What does x^2-x become or how much does x^2 increase when x becomes $x+1$,

$$x+a,$$
$$x+\Delta x\,?$$

are mere variations on the "four rules," but have a visible purpose.

The converse problems of summation, and of constructing the first column so that the second column may be of given form, and the problem of the relative importance of various terms must be introduced cautiously.

The fact that in any example the difference between any two terms in the first column is equal to the sum of the intervening terms in the second column is easily realised, and may be illustrated by obtaining the formula for the sum of a geometric progression

y	Δy
a	
	$a(r-1)$
ar	
	$ar(r-1)$
ar^2	$\vdots$
$\vdots$	$ar^{n-1}(r-1)$
ar^n	

so $$(r-1)\times(a+ar+\ldots+ar^{n-1})=ar^n-a.$$

The notation $\Sigma(\Delta y)=y_n-y_1$ to express these and similar facts may be introduced by means of special cases, illustrated by graphs.

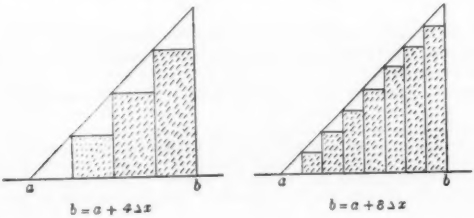
The labour of completing a column of differences may be divided up among a class, with good effects on the desire for accuracy. It seems probable that a little of this rather heavy algebraic summation is necessary to the comprehension of the easier and more general methods.

The next step is to see that it will be well to indicate by the notation exactly how many terms are included in the summation. We agree to do so by indicating the range over which x extends. The graphical accompaniment helps to make this clear.

It is quite possible, by a discussion of what happens if the journey from a to b is made by "ever so many" steps each "ever so small," to end with the formula

$$\int_a^b x\,dx=\frac{b^2-a^2}{2},$$

every step of the work being illustrated by a diagram.



The notion that the accurate formula

$$\Sigma(\Delta u)=u_n-u_1$$

leads when the steps become "ever so small" to the equally accurate formula

$$\int_0^1 du=u_1-u_0, \text{ say,}$$

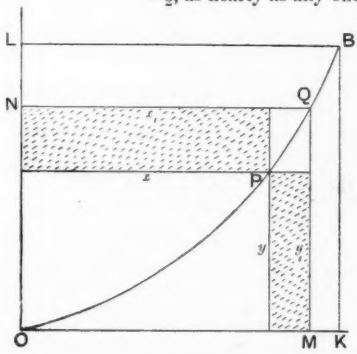
may be developed cautiously with abundant illustrations; such as $\frac{1}{2}r^2\Delta\theta$ as element of area, $2\pi r\Delta r$ as ring-shaped element of area in a circle; $\pi y^2\Delta x$ as coin-shaped element of volume in a solid of revolution.

The following brief outline of a method of finding the area of the parabola $y=x^2$ may be helpful, as a collateral illustration.

$$\frac{\text{Rectangle } PM}{\text{Rectangle } PN}=\frac{y(x_1-x)}{x(y_1-y)}=\frac{x^2(x_1-x)}{x(x_1^2-x^2)}$$

$$=\frac{x}{x_1+x}=\frac{1}{1+\frac{x_1}{x}}$$

$$=\tfrac{1}{2}, \text{ as nearly as any one pleases.}$$



As this holds for each pair of rectangles, the ratio of the area *OBK* to *OBL* is $\frac{1}{2}$, and hence the area *OBK* is one-third of that of the rectangle *OBLK*.

With boys of fifteen the argument must be put in a broad manner—a manner that might be stigmatised as "soppy" at a later period.

Of course $y=x^n$ can be dealt with in the same way, and also $y^r=x$. [n, r, s all positive integers.]

$$\frac{\text{Rectangle } PM}{\text{Rectangle } PN}=\frac{y(x_1-x)}{x(y_1-y)}$$

$$=\frac{y}{x}\cdot\frac{x_1^s-x^s}{x_1^{s-1}+xx_1^{s-2}+x^{s-1}}\times\frac{y_1^{r-1}+yy_1^{r-2}\ldots+y^{r-1}}{y_1^r-y^r}$$

$$=\frac{y}{x}\cdot\frac{y_1^{r-1}+yy_1^{r-2}\ldots+y^{r-1}}{x_1^{s-1}+xx_1^{s-1}\ldots+x^{s-1}}$$

$$=\frac{ry^r}{sx^s}, \text{ as nearly as any one pleases,}$$

$$=\frac{r}{s} \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

Reference is made to articles by Dr. Nunn in the *Mathematical Gazette*, Dec. 1910 and Jan. 1911.

All this preliminary work might, it is submitted, be embodied, if treated lightly, in the traditional school work without any great disturbance. It is in the main elementary algebra and not the calculus.

This is emphasised because of a danger which must be kept in mind. We are all familiar with the behaviour of a heavy body acted on by a disturbing force. First, there is a time-lag—the body is slow to respond. Secondly, there is a so-called inertia effect—the body will indicate a displacement sometimes vastly in excess of that which can be statically maintained by the pressure applied. Teachers of mathematics are persons of weight, and are not exempt from the concomitants of their qualities.

For instance—it is not without significance that teachers of mathematics, as a body, are beginning to concern themselves with questions of practical work, just as a note of warning is being sounded by some of our most experienced science teachers as to the futility of spending months and months and months in *mere* weighing and *mere* measuring. There is the time-lag. The inertia effect will appear if, in deference to the demand that the calculus shall be made intelligible, this preliminary work, which is intended to lead up to the calculus, is cultivated to excess and allowed to take the place of the calculus.

The engineer's demand for the calculus arises in part from his discontent with tentative graphical methods: and he will tell us presently that it is a pity to be "plus royaliste que le roi."

Coming then to the formal beginning of the calculus, the differential calculus naturally originates in two ways: by an analysis of the common notion of "variable velocity" and by the consideration of the problem of drawing a tangent to a curve.

We have Lord Kelvin's authority for saying that a differential coefficient is primarily a velocity.

From several such equations as

$$s = 16t^2, \quad s = 12t - t^2,$$

the notions of $\frac{\Delta s}{\Delta t}$ as average velocity and $\frac{ds}{dt}$ as velocity at an instant are acquired.

C. S. JACKSON.

(To be continued.)

QUERIES.

(85) A question of notation.

Ought we to write 0·125 or ·125 for one-eighth expressed as a decimal? What is the origin of the former method, and what are deemed to be its advantages?

The principle, perhaps the most important one in all arithmetic, of keeping figures of the same local value in the same vertical column does not seem to be involved.

C. S. J.

(86) In the English edition of David Gregory's *Elements of Astronomy* (Vol. II., London, MDCCXV., pp. 732-733) occurs the following famous problem of Newton's *Principia* (1687, Lemma 27, Book I.): "Four right lines given in position, lying in the same plane, to draw a fifth which shall be cut by these four, in three parts, having a given ratio and a given order among themselves." It is stated that the construction given "is the Invention of the famous Geometrician Sir Christopher Wren." Newton concludes his Lemma with: "Problematis hujus solutiones alias Wrennus et Wallisius olim ex cogitaverunt."

When and where were these solutions of Wren and Wallis published?

R. C. ARCHIBALD.

(87) A point P has given tripolar coordinates (PA, PB, PC). A point Q has given barycentric coordinates. Determine the distance PQ. W. G.

(88) *A Mathematical Recreation.*—With the digits 1, 2, ... n, express the consecutive numbers from 1 upwards as far as possible, say to p: four and only four digits, all different, being used in each number, and the notation of the denary scale (including decimals), as also algebraical sums, products, and powers being allowed. If the use of the symbols for square roots and factorials (repeated if desired a finite number of times) is also allowed, the range can be considerably extended, say, to q consecutive integers. If $n=4$, I make $p=88$, $q=264$; if $n=5$, $p=231$, $q=790$; with the digits 0, 1, 2, 3, $p=36$, $q=40$. The problem is not easy, and these limits may be too low.

W. W. R. B.

ANSWER TO QUERY.

[**83, p. 379, vol. vi.**] This we take to mean the chance that in an ordinary pack of cards, one king and one queen *at least* may come next each other *irrespective of suits.*

If, in considering the number of arrangements, we regard the kings as all alike, and the queens all alike, and the other 44 cards as indistinguishable from each other,

$$\text{the total number of arrangements} = \frac{52 \cdot 51 \ldots 45}{\underline{|4} \times \underline{|4}}. \quad \ldots\ldots\ldots\ldots(1)$$

To calculate the number of arrangements in which *no* king and queen come together, consider first the relative positions of kings and queens, ignoring the other cards. It is obvious that kings and queens may alternate any number of times from 1 to 7. If in any particular arrangement there are r alternations, there will be r places in which one plain card *at least* must be inserted. Supposing just r to have been inserted in these places, the remaining $44-r$ plain cards may be distributed in *any* way in the 9 spaces separated by the 8 kings and queens. The number of ways of thus distributing $44-r$ like things into 9 groups (with any number from 0 upwards in each) = the number of permutations of $52-r$ things, of which $44-r$ are of one kind and 8 of another (the latter just serving to separate the former into the number of groups in question, where any group or groups may be zero)

$$= {}_{52-r}C_8. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But if $r=2m+1$, the number of arrangements of kings and queens which will give r alternations $= 2\left(\frac{\underline{|3}}{\underline{|m}\,\underline{|3-m}}\right)^2$, there being $m+1$ groups of one or more kings, and $m+1$ groups of one or more queens; while we may start with either, hence the factor 2. Similarly if $r=2m$, the number $= 2\frac{\underline{|3} \times \underline{|3}}{\underline{|m}\,\underline{|3-m}\,\underline{|m-1}\,\underline{|4-m}}$, there being now $m+1$ groups of the one and m of the other. The numbers thus given for values of r from 1 to 7 respectively are 2, 6, 18, 18, 18, 6, 2; and therefore by (2) the total number of arrangements in which no king and queen come together

$$= 2_{51}C_8 + 6_{50}C_8 + 18({}_{49}C_8 + {}_{48}C_8 + {}_{47}C_8) + 6_{46}C_8 + 2_{45}C_8. \quad \ldots\ldots\ldots\ldots(3$$

Dividing by the number first found, (1), we have for the chance that *no* king and queen come together

$$\frac{2}{70} \times \frac{44}{52} + \frac{6}{70} \times \frac{44 \cdot 43}{52 \cdot 51} + \frac{18}{70} \frac{44 \cdot 43 \cdot 42}{52 \cdot 51 \cdot 50} + \ldots + \frac{2}{70} \frac{44 \ldots 38}{52 \ldots 46}.$$

Worked out arithmetically, the result $=\frac{300684703}{585307450}=\frac{20}{39}$ nearly. Therefore, the chance that at least one king and queen may come together $=\frac{284622747}{585307450}=\frac{19}{39}$ nearly. It is almost an even chance.

PERCY J. HEAWOOD.

MATHEMATICAL NOTES.

411. [A. 1. a.] *Fractional and negative values of n in Arithmetical Progressions.*

These are often puzzling to learners, who are sometimes led to suppose that no meaning can be given to them.

In the formula $$s=\frac{n}{2}\{2a+(n-1)d\},$$

replace n by $\frac{p}{q}$, where p and q are positive integers; thus

$$s=\frac{p}{2q}\left\{2a+\left(\frac{p}{q}-1\right)d\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This may be written

$$s=\frac{p}{2}\left\{2\left(\frac{a}{q}-\frac{d}{2q}+\frac{d}{2q^2}\right)+(p-1)\frac{d}{q^2}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now (2) is the formula for the sum of p terms of an A.P. whose first term is $\frac{a}{q}-\frac{d}{2q}+\frac{d}{2q^2}$ and common difference $\frac{d}{q^2}$; hence this is also the meaning of equation (1).

The case when n is a negative integer (say $n=-p$) may be worked out separately, or we can deduce it from (2) by putting $q=-1$; thus

$$s=\frac{p}{2}\{2(d-a)+(p-1)d\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

i.e. the sum of p terms of the A.P. whose first term is $d-a$ and common difference d.

If $d-a$ happens to be one of the terms of the given series (continued backwards if necessary), this gives us the same information about that series as we obtain in the simple case when *both* roots of the quadratic are positive integers.

E.g. how many terms of the series 3, 5, 7, ... amount to 8?

The quadratic for n gives $n=2$ or -4. We see by equation (3) that the value -4 is to be interpreted as 4 terms of the series whose first term is $2-3$ and common difference 2.

These terms are -1, 1, 3, 5 (another portion of the given series), their sum being 8 as before.

The condition that $d-a$ should be a term of the given series is easily found.

G. OSBORN.

412. [K¹. 2. b, c.] *Proof of Feuerbach's theorem.*

In the triangle ABC let D, E, F be the middle points of the sides, X, Y, Z the points of contact of the inscribed circle, so that

$$DX=\tfrac{1}{2}(c-b), \quad EY=\tfrac{1}{2}(a-c), \quad FZ=\tfrac{1}{2}(b-a).$$

[Lemma] If a point P moves so that $DP:EP::DX:EY$, it describes a circle U orthogonal to any circle through D, E, such as DEF, and also orthogonal to the circle XYZ, because DX, EY are the tangents from D, E to XYZ. (For proofs *v.* below.)

If a point Q moves so that $EQ:FQ::EY:FZ$, then it describes a circle V orthogonal to DEF and XYZ.

Let a, b, c be the order of magnitude of the sides. Then there is a point J on the circle DEF but not on the arc DEF, such that

$$DJ : JF :: c-b : b-a,$$

and EJ is proportional to $c-a$, since

$$b \,.\, EJ = a \,.\, DJ + c \,.\, JF \quad \text{(Ptolemy)}.$$

Thus, J is common to the circles U, V and DEF. Hence U, V must touch at J, both being orthogonal to DEF. Since DEF, XYZ are both orthogonal to U, V, they must touch one another at J.

Hence *the inscribed and nine-point circles touch at the point J whose distances from D, E, F are as $b-c$, $c-a$, $a-b$.*

If P moves so that $AP : PB :: t_1 : t_2$, where t_1, t_2 are the tangents from A, B to a circle Ω, then the circle which is the locus of P cuts Ω orthogonally.

First proof. Through A, B describe a circle X not meeting Ω. (This is possible because A, B are both outside or both inside Ω.)

Let L, M be the limiting points of X and Ω. Then

$$\frac{AL}{BL} = \frac{t_1}{t_2} = \frac{AM}{BM}.$$

Hence the locus of P passes through L, M and cuts orthogonally any circle coaxial with X and Ω. Q.E.D.

Second proof. By inversion from A or B. For we know that if O is centre of inversion, A' the inverse of A, Ω' the inverse of Ω, then

(Power of A *w.r.t.* Ω)/OA^2=(Power of A' *w.r.t.* Ω')/(Power of O *w.r.t.* Ω').

Third proof. Let the locus cut Ω in Q. Produce AQ, BQ to meet Ω again in F, G.

Then
$$\frac{AQ^2}{BQ^2} = \frac{t_1^2}{t_2^2} = \frac{AQ \,.\, AF}{BQ \,.\, BG};$$

$$\therefore \frac{AQ}{BQ} = \frac{AF}{BG}, \quad \text{and } AB, FG \text{ are parallel.}$$

Hence the circles ABQ and Ω touch at Q. But the locus is orthogonal to ABQ at Q, and therefore to Ω.

Fourth proof. $\dfrac{AP}{t_1} = \dfrac{BP}{t_2}$. $\therefore$ A, B lie on a circle coaxial with Ω and the pt. P: the locus of P is that of the limiting pts. of coaxial systems containing Ω and different circles through A, B: these limiting pts. must lie on the circle which cuts orthogonally Ω and two circles through A, B. A. C. Dixon.

REVIEWS.

The Twisted Cubic, etc. By P. W. Wood. (Camb. Math. Tracts, No. 14.) 2s. 6d. 1913. (Camb. Univ. Press.)

The object of this tract is to supply a connected analytical treatment of the properties of the space curve of order three. No previous acquaintance with the curve is assumed, and analytical methods have been used throughout in preference to synthetical.

Projective properties are discussed by means of the parametric expressions $(\theta^3, \theta^2, \theta, 1)$, and $\left(\frac{1}{\theta-a}, \frac{1}{\theta-b}, \frac{1}{\theta-c}, \frac{1}{\theta-d}\right)$. The latter form is of special use when four or five fixed points on the curve are involved, and in the former the choice of $\frac{1}{1}$ as the fourth coordinate makes some of the expressions more symmetrical. Some knowledge of the theory of screws is assumed.

The author is specially interested in the relation of the curve to binary quartics; he has not found space for references to the involution of binary cubics or to the quadriquadric equation, which present themselves quite as naturally when the relations of a straight line and the curve are considered, and in fact define the straight line without ambiguity, whereas the quartic does not distinguish

between the two lines that meet four given tangents. Moreover, in discussing the Hessian of the quartic, the author does not point out that one of the lines defined by it is the locus of harmonic poles referred to on p. 61.

The second half of the tract contains a study of the metrical properties of the cubical hyperbola, chiefly in reference to the parallelopiped which has the asymptotes as three edges. The only mistake in the book seems to be on p. 71, where it is stated that the condition for a rectangular hyperbola is linear in the coefficients of the tangential equation.

It seems strange that Salmon should only be mentioned once, and only in relation to a locus which, as he pointed out, is not a twisted cubic.

These are small matters, and do not affect the fact that the book is an able and excellent piece of work, showing great care on the part of both author and printer. The following are some of the items that should tempt readers:

On p. 26, the theorem that if two tetrahedra are inscribed in the curve their faces osculate another cubic.

On pp. 26-29, the construction of four tetrahedra such that each vertex of any one lies on a face of each of the others.

On p. 43, the symmetrical parametric representation used for the cubical hyperbola.

On p. 69, the theorem that in a rectangular cubical hyperbola, if an inscribed tetrahedron has an orthocentre, the orthocentre lies on the curve. A. C. DIXON.

The Principle of Least Action. By PHILIP E. B. JOURDAIN. Pp. 83. Price 1s. 6d. 1913. (Open Court Publishing Co.)

This is a reissue of three articles contributed to the *Monist* by Mr. Jourdain. The first deals with Maupertuis' enunciation of the principle, and with the contemporary discussions which Euler and others contributed. It is well known that the discussion involved controversy doubly or trebly acute, because not only mathematics, but metaphysical theology and questions of priority were involved. Mr. Jourdain refers to every writer on the subject, and his work will be indispensable to subsequent inquirers. Our impression is that in his account of the controversy he is a little hard on Voltaire, whose anger was set aflame, as Lord Morley says, not by intellectual vapidity, but by what he counted gross wrong. The conduct of Maupertuis towards König struck Voltaire and all men since as tyrannical, unjust and childish. But if we can send any English reader to Dr. Akakia himself, we shall have earned that reader's undying gratitude, and may leave him to form his own opinion. He will be interested to find Benjamin Robins, a name unknown to most Englishmen, recognised abroad as one of the leaders of European thought.

The second essay deals with the relation of the principle to Hamilton's principle of stationary variation, and the corresponding theorems of the calculus of variations. The remaining essay, in the author's words, is critical and expository. The relation of the principle of least action to the principle of *vis viva* is analysed. The very numerous bibliographical references, all of which have been actually consulted, will give some idea of the enormous labour of Mr. Jourdain's task, a labour which must entitle him to the gratitude of all workers in this particular field, whether their interest is mainly in the history, the mechanics, or the analysis. C. S. J.

"Squaring the Circle": a History of the Problem. By E. W. HOBSON. Pp. 57. 3s. 1913. (Camb. Univ. Press.)

The Squaring of the Circle owes its importance to three facts: the simple, everyday ideas used in its statement; its immense age, and the amount of labour which has been devoted to it; and the completeness of its solution, at last achieved by the help of far-fetched analysis.

Chap. I. of Professor Hobson's tract contains a careful discussion of what is meant by a Euclidean construction, and a general survey of the history of the problem; Chap. II., after some prehistoric peeps, is mainly devoted to the geometric methods of in- and circumscribed polygons, from Archimedes to Huyghens, a period which includes Ludolph's arithmetical feats. Chap. III. shows how the invention of the differential calculus led to the use of infinite series for π, and to results whose accuracy culminates in Shanks' 707 decimal places; and the last chapter tells how a final end was put to the labours of those who sought an accurate arithmetical value of π by Lambert's proof of its irrationality,

and of those who sought an accurate geometrical construction, by Lindemann's proof of its transcendence—of the sane ones, that is. After a digression on the algebraic formulation of Euclidean construction, a proof is given of the transcendence of π, "founded on that of Gordan," but both fuller and clearer than any yet published.

The abundant humour of the subject has been most happily described by De Morgan; and by publishing these lectures, which deal seriously with the famous problem, Professor Hobson has filled a real gap in English mathematical literature.

H. P. H.

The Laws of Thermodynamics. By W. H. MACAULAY. Pp. viii+71. 3s. net. 1913. (Camb. Univ. Press.)

This is the second of the *Cambridge Engineering Tracts*, and it is a masterly presentation of the subject for the beginner, and, as far as may be within its slender limits of space, from the point of view of the engineer. The aim is "to provide an accurate and connected account of the fundamental principles of thermodynamics, combined with a sketch of methods applying the theory in special cases, to supplement technical books on the subject." The wise beginner will keep it at his side for reference as his acquaintance with the subject proceeds. Perfect differentials are first explained and the first law is expounded. Particular types of substance are next dealt with, *e.g.* superheated vapour, throttled vapour, and perfect gas. A useful historical note is appended to the chapter on the second law. Entropy forms the subject-matter of the next two chapters, which conclude with an account of the electromotive force of a storage battery. In the final pages the theory of entropy is extended to the case of any number of independent variables. The author handles his subject throughout in a manner both concise and clear.

Key to A New Algebra. By S. BARNARD and J. M. CHILD. Vol. II., containing Parts IV., V. and VI. Pp. 447-915. 1913. (Macmillan.)

This Key not merely contains model solutions of all the problems, but alternative solutions at times are given, and notes explaining the principles involved are, when necessary, supplied. It will be a useful addition to the library of the student who has no one at his side to guide his progress, and there will no doubt be many teachers who will find that such a volume will save them considerable time and worry.

A Text-Book of Elementary Statics. By R. S. HEATH. Pp. xii+284. 4s. 6d. 1912. (Clarendon Press.)

This is a useful introduction to Statics and its applications so far as it can be applied without the aid of the Calculus. The treatment is both analytical and geometrical. The examples worked out are very judiciously selected, and deal almost entirely with the machines of practical life—bicycles, steam-engines, real pulleys and weighing machines, the chemical balance, overhauling, and the like. Graphical theory is not forgotten. The examples throughout are a mixture of the old and new styles. Chapter VII. may be taken as a fair sample of the arrangement. Here we have in succession: conditions of equilibrium of plane systems of forces, analytical reduction of a system of coplanar forces, general reduction of any system, reduction of a system to two forces, and so on, followed by worked-out examples on the rowing-boat, pressure on pedals and tension of chains of bicycle upon inclined plane, the step-ladder, the driving couple of a steam-engine, the tripod, and the casement window. The clear and unaffected style of the author will commend itself to the reader.

Graphical Methods. By CARL RUNGE. Pp. vii+148. 6s. 6d. net. 1912. (Columbia University Press.)

"L'Algèbre n'est qu'une géométrie écrite, la géométrie n'est qu'une algèbre figurée."

Graphical methods are no doubt a most valuable means of giving to the student a clear idea in concrete form of the symbolism which, without their aid, may remain more or less of a mystery to the minds of the majority.

Prof. Runge passes in review the various methods that have been contrived by ingenious minds to meet special needs, and then proceeds to generalise them "so as to facilitate their application in any problem with which they are mathe-

matically connected." Beginning with graphical calculation as applied to arithmetic, integral and linear functions, and complex numbers, he proceeds from the graphical representation of functions of one or more independent variables to the graphical methods in the differential and integral Calculus. He keeps in sight the imperative necessity—especially in the case of the young engineer and others who will use graphs in the ordinary routine of their daily work—of acquiring facility in the application of the fundamental ideas. "You might as well try to learn piano-playing only by attending concerts as to learn the graphical methods only through lectures." The book is well conceived, and should prove extremely suggestive to those who master its contents.

The Elements of Descriptive Astronomy. By E. O. TANCOCK. Pp. 110. 2s. 6d. net. 1913. (Clarendon Press.)

As the sub-title indicates, this little book is "A simple account of the Celestial Bodies and their motions." The author suggests that it will be useful for such elementary examinations as those taken by the Boy Scouts. The explanations explain and the illustrations and diagrams are deserving of special mention. We are inclined to think that this unpretentious little introduction to the marvels of the sky will induce many a boy to seek for further information, and will develop a taste for practical work in those who are fortunate enough to be taken through this course.

Marsh's Mathematics Work-Book. Designed by H. W. MARSH. 3s. net. 1913. (Wiley, Chapman & Hall.)

Mr. Marsh provides each boy with some 250 pages of good writing paper in a strong cover, which looks as if it would stand years of wear. Large paper fasteners attach the sheets to this cover, so that they are easily detachable. A few Daily Record sheets are supplied, comprising, after spaces for Name, Subject, Course, Term, divisions headed: Date, Paragraph, Page, Problem or Theorem, Remarks, Hours Outside Preparation, Grand Total in hours, Subjects Totals in hours. This is followed by the following: I certify that the above is a true record of the time actually devoted by me to the outside individual preparation of the work specified. To this the student's signature is to be appended, and he is warned on each Record that "well-arranged time is the surest sign of a well-arranged mind." Sheets pasted on the inside of the covers give minute directions for the use of the book, accompanied by many useful hints, *e.g.* "as soon as possible learn to draw a light, smooth, draftsman's line."

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2	copies of *Gazette*	No. 2 (very important).
1 or 2	„ „	No. 8.
2 or 3	copies of Annual Report	No. 11 (very important).
1 or 2	„ „	Nos. 10, 12 (very important).
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